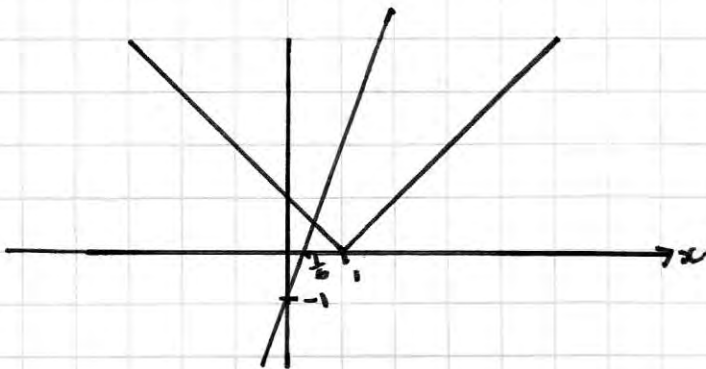


1. Find the set of values for which

$$|x-1| > 6x-1.$$

(5)



Intersection with reflected
part of modulus

$$\therefore x-1 = 1-6x$$

$$7x = 2 \quad x = \frac{2}{7}$$

$$\therefore x < \frac{2}{7}$$

2. (a) Find the general solution of the differential equation

$$t \frac{dv}{dt} - v = t, \quad t > 0$$

and hence show that the solution can be written in the form $v = t(\ln t + c)$, where c is an arbitrary const. (6)

(b) This differential equation is used to model the motion of a particle which has speed $v \text{ m s}^{-1}$ at time $t \text{ s}$. When $t = 2$ the speed of the particle is 3 m s^{-1} . Find, to 3 sf, the speed of the particle when $t = 4$. (4)

$$\frac{dv}{dt} - \frac{1}{t} v = 1 \quad \text{If } f(t) = e^{\int -\frac{1}{t} dt} = (e^{\ln t})^{-1} = \frac{1}{t}$$

$$\frac{1}{t} \frac{dv}{dt} - \frac{1}{t^2} v = \frac{1}{t} \Rightarrow \frac{d}{dt} \left(\frac{v}{t} \right) = \frac{1}{t} \Rightarrow \frac{v}{t} = \int \frac{1}{t} dt \Rightarrow \frac{v}{t} = \ln t + c$$

$$\therefore v = t \ln t + ct$$

$$t=2, v=3 \quad 3 = 2 \ln 2 + 2c \quad c = \frac{3}{2} - \ln 2$$

$$\therefore v = t \ln t - t \ln 2 + \frac{3}{2} t = t \ln \left(\frac{t}{2} \right) + \frac{3}{2} t$$

$$t=4 \quad v = 4 \ln 2 + 6 \quad \approx \underline{\underline{8.77}}$$

3. (a) Show that $y = \frac{1}{2}x^2e^x$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x. \quad (4)$$

(b) Solve the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$.

given that at $x=0$, $y=1$ and $\frac{dy}{dx} = 2$.

(9)

$$y = \frac{1}{2}x^2e^x$$

$$y' = \frac{1}{2}x^2e^x + xe^x = (\frac{1}{2}x^2 + x)e^x$$

$$y'' = (\frac{1}{2}x^2 + x)e^x + (x+1)e^x = (\frac{1}{2}x^2 + 2x + 1)e^x$$

$$y'' = (\frac{1}{2}x^2 + 2x + 1)e^x$$

$$-2y' = (-x^2 - 2x)e^x +$$

$$+y = (\frac{1}{2}x^2)e^x$$

$$e^x = e^x \quad \#$$

$\therefore y = \frac{1}{2}x^2e^x$ is the particular integral solution to the 2nd order dy/dx equation.

b) $y = Ae^{mx}$

$$y' = Ame^{mx}$$

$$y'' = Am^2e^{mx}$$

$$y'' - 2y' + y = 0$$

$$Ae^{mx}(m^2 - 2m + 1) = 0$$

$$\neq 0$$

$$= 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1 \text{ RR}$$

$$y_{cf} = (A+Bx)e^x$$

$$\therefore y = (A+Bx + \frac{1}{2}x^2)e^x$$

$$x=0, y=1 \Rightarrow \underline{1=A}$$

$$y = (1+Bx + \frac{1}{2}x^2)e^x$$

$$y' = (1+Bx + \frac{1}{2}x^2)e^x + (B+x)e^x$$

$$x=0 \quad y'=2$$

$$2 = 1+B \quad \therefore \underline{B=1}$$

$$\therefore y = (1+x + \frac{1}{2}x^2)e^x$$

7

4. The curve C has polar equation $r = 3a \cos \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

The curve D has polar equation $r = a(1 + \cos \theta)$, $-\pi \leq \theta < \pi$. Given that a is a positive constant,

(a) sketch, on the same diagram, the graphs of C and D, indicating where each curve cuts the initial line.

(4)

The graphs of C intersect at the pole O and at the points P and Q.

(b) Find the polar coordinates of P and Q.

(3)

(c) Use integration to find the exact area enclosed by the curve D and the lines $\theta = 0$ and $\theta = \frac{\pi}{3}$

(7)

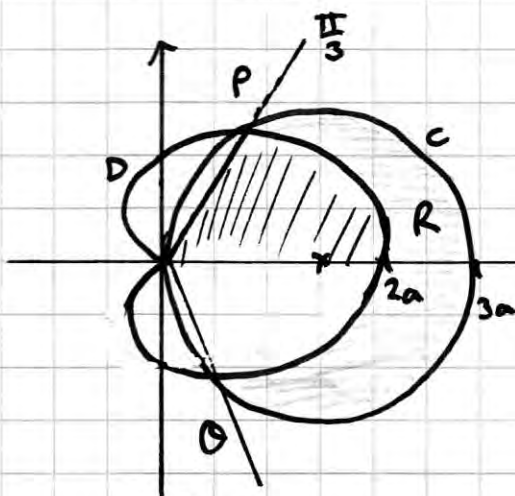
The region R contains all points which lie outside D and inside C.

Given that the value of the smaller area enclosed by the curve C and the line $\theta = \frac{\pi}{3}$ is

$$\frac{3a^2}{16} (2\pi - 3\sqrt{3}),$$

(d) show that the area of R is πa^2 .

(4)



$$\begin{aligned} 3a \cos \theta &= a(1 + \cos \theta) \\ 2 \cos \theta &= 1 \Rightarrow \cos \theta = \frac{1}{2} \\ \theta &= \frac{\pi}{3}, -\frac{\pi}{3} \end{aligned}$$

$$\begin{aligned} \theta = 0 \quad r &= a(1 + (\frac{1}{2})) = \frac{3}{2}a \\ P(\frac{3}{2}a, \frac{\pi}{3}) \quad Q(\frac{3}{2}a, -\frac{\pi}{3}) \end{aligned}$$

$$\begin{aligned} \text{c) Area} &= \frac{1}{2}a^2 \int_0^{\pi/3} (1 + \cos \theta)^2 d\theta = \frac{1}{2}a^2 \int_0^{\pi/3} 1 + 2\cos \theta + (\frac{1}{2} + \frac{1}{2}\cos 2\theta) d\theta \\ &= \frac{1}{2}a^2 \int_0^{\pi/3} \frac{3}{2} + 2\cos \theta + \frac{1}{2}\cos 2\theta d\theta = \frac{1}{4}a^2 \int_0^{\pi/3} 3 + 4\cos \theta + \cos 2\theta d\theta \\ &= \frac{1}{4}a^2 [3\theta + 4\sin \theta + \frac{1}{2}\sin 2\theta]_0^{\pi/3} = \frac{1}{4}a^2 [(\pi + 2\sqrt{3} + \frac{\sqrt{3}}{4}) - (0)] \\ &= \frac{1}{4}a^2 (\pi + \frac{9\sqrt{3}}{4}) = \frac{1}{16}a^2 (4\pi + 9\sqrt{3}) \quad \square \end{aligned}$$

$$\text{d) } R = 2 \times \left[\frac{\pi (\frac{3}{2}a)^2}{2} - \frac{1}{16}a^2 (4\pi + 9\sqrt{3}) - \frac{3}{16}a^2 (2\pi - 3\sqrt{3}) \right]$$

$$R = \frac{9\pi}{4}a^2 - \frac{1}{8}a^2 (4\pi + 9\sqrt{3} + 6\pi - 9\sqrt{3}) = \left(\frac{9\pi}{4} - \frac{10\pi}{8} \right) a^2 = \frac{\pi a^2}{2}$$

5. Using algebra, find the set of values of x for which $2x - 5 > \frac{3}{x}$.

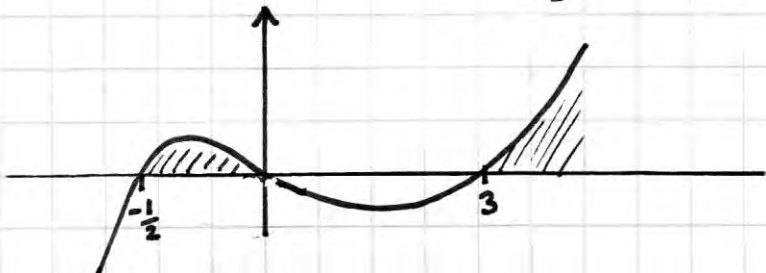
(7)

$$(2x - 5)x^2 > \frac{3x^2}{x} \Rightarrow 2x^3 - 5x^2 - 3x > 0$$

$$x(2x^2 - 5x - 3) > 0$$

$$x(2x + 1)(x - 3) > 0$$

$\begin{matrix} 0 & -\frac{1}{2} & 3 \end{matrix}$



$$-\frac{1}{2} < x < 0 \quad \text{or} \quad x > 3$$

6. (a) Find the general solution of the differential equation

$$\cos x \frac{dy}{dx} + (\sin x)y = \cos^3 x.$$

(6)

- (b) Show that, for $0 \leq x \leq 2\pi$, there are two points on the x -axis through which all the solution curves for this differential equation pass. (2)

- (c) Sketch the graph, for $0 \leq x \leq 2\pi$, of the particular solution for which $y = 0$ at $x = 0$. (3)

$$\frac{dy}{dx} + (\tan x)y = \cos^2 x \quad \text{If } f(x) = e^{\int \tan x dx} = e^{\ln |\sec x|} = \sec x$$

$$\sec x \frac{dy}{dx} + (\sec x \tan x)y = \cos x$$

$$\frac{d}{dx}(y \sec x) = \cos x \Rightarrow y \sec x = \int \cos x dx = \sin x + c$$

$$\therefore y = \sin x \cos x + c \cos x$$

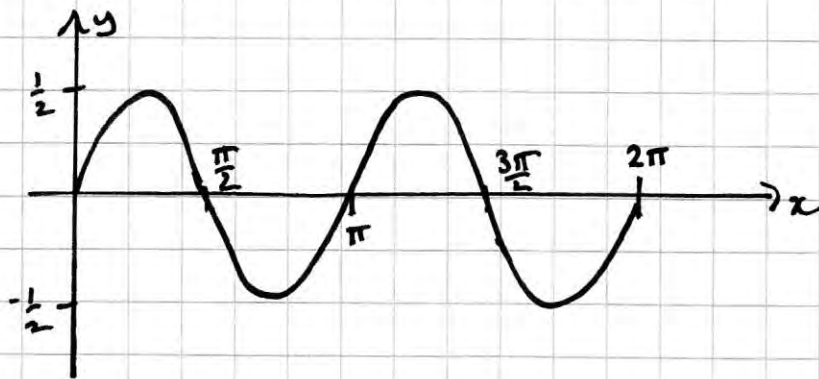
$$b) \quad \sin x \cos x + c \cos x = 0 \Rightarrow \cos x (\sin x + c) = 0$$

$$\cos x = 0$$

$$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$(0, 0) \Rightarrow 0 = 0 + c \therefore c = 0$$

$$\therefore y = \sin x \cos x \quad y = \frac{1}{2} \sin 2x$$



7. (a) Find the general solution of the differential equation

$$2 \frac{d^2 y}{dt^2} + 7 \frac{dy}{dt} + 3y = 3t^2 + 11t. \quad (8)$$

(b) Find the particular solution of this differential equation for which $y = 1$ and $\frac{dy}{dt} = 1$ when $t = 0$. (5)

(c) For this particular solution, calculate the value of y when $t = 1$. (1)

$$\begin{aligned} y &= Ae^{mt} \\ y' &= Ame^{mt} \\ y'' &= Am^2 e^{mt} \end{aligned} \quad \begin{aligned} 2y'' + 7y' + 3y &= 0 \\ \therefore Ae^{mt}(2m^2 + 7m + 3) &= 0 \\ \neq 0 \quad = 0 \quad (2m+1)(m+3) &= 0 \quad m = -\frac{1}{2} \quad m = -3 \end{aligned} \quad y_{\text{hom}} = Ae^{-3t} + Be^{-\frac{1}{2}t}$$

$$\begin{aligned} y &= a + bt + ct^2 \\ y' &= b + 2ct \\ y'' &= 2c \end{aligned} \quad \begin{aligned} + 3y &= 3a + 3bt + 3ct^2 \\ + 7y' &= 7b + 14ct \\ + 2y'' &= 4c \end{aligned} \quad \begin{aligned} 3t^2 + 11t &= (3a + 7b + 4c) + (3b + 14c)t + 3ct^2 \end{aligned}$$

$$\therefore 3c = 3 \quad \therefore c = 1 \quad 3b + 14 = 11 \quad \therefore b = -1$$

$$3a - 7 + 4 = 0 \quad \therefore a = 1$$

$$y_{\text{PI}} = 1 - t + t^2$$

$$\therefore y = Ae^{-3t} + Be^{-\frac{1}{2}t} + 1 - t + t^2$$

$$y' = -3Ae^{-3t} - \frac{1}{2}Be^{-\frac{1}{2}t} - 1 + 2t$$

$$t=0, y=1 \quad 1 = A + B + 1 \quad \therefore A + B = 0$$

$$t=0, y'=1 \quad 1 = -3A - \frac{1}{2}B - 1 \quad \underline{6A + B = -4}$$

$$-5A = 4 \quad A = -\frac{4}{5} \quad B = \frac{4}{5}$$

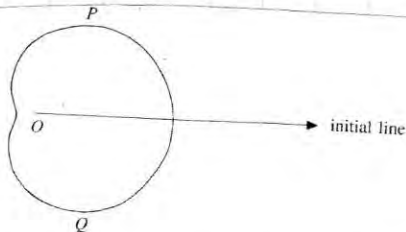
$$\therefore y = \frac{4}{5}e^{-\frac{1}{2}t} - \frac{4}{5}e^{-3t} + 1 - t + t^2$$

$$t=1 \quad y = \frac{4}{5}e^{-\frac{1}{2}} - \frac{4}{5}e^{-3} + 1 = 1.445 \dots$$

Figure 1

The curve C shown in Fig. 1 has polar equation

$$r = a(3 + \sqrt{5} \cos \theta), \quad -\pi \leq \theta < \pi.$$



(a) Find the polar coordinates of the points P and Q where the tangents to C are parallel to the initial line. (6)

The curve C represents the perimeter of the surface of a swimming pool. The direct distance from P to Q is 20 m.

(b) Calculate the value of a . (3)

(c) Find the area of the surface of the pool. (6)

$$a) \frac{dy}{d\theta} = 0 \quad y = r \sin \theta = a(3 + \sqrt{5} \cos \theta) \sin \theta$$

$$\frac{dy}{d\theta} = a(3 + \sqrt{5} \cos \theta) \cos \theta + a(-\sqrt{5} \sin \theta) \sin \theta = 0$$

$$\Rightarrow \sqrt{5} \sin^2 \theta = 3 \cos \theta + \sqrt{5} \cos^2 \theta$$

$$\Rightarrow \sqrt{5}(1 - \cos^2 \theta) = 3 \cos \theta + \sqrt{5} \cos^2 \theta$$

$$\Rightarrow 2\sqrt{5} \cos^2 \theta + 3 \cos \theta - \sqrt{5} = 0$$

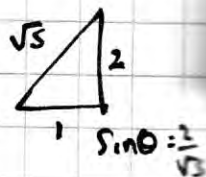
$$(\sqrt{5} \cos \theta - 1)(2 \cos \theta + \sqrt{5}) = 0$$

$$\cos \theta = \frac{1}{\sqrt{5}} \quad \cos \theta = -\frac{\sqrt{5}}{2} \text{ no solutions}$$

$$\theta = 1.107^\circ, -1.107^\circ$$

$$r = a(3 + \sqrt{5}(\frac{1}{\sqrt{5}})) = 4a$$

$$P(4a, 1.107) \quad Q(4a, -1.107)$$



$$b) \begin{array}{c} \text{Diagram: A right-angled triangle with vertices O, P, Q. The hypotenuse PQ is the direct distance between P and Q. The side OP is labeled 4a. The side OQ is labeled 4a \sin \theta = \frac{8}{\sqrt{5}}a. The angle at O is labeled \theta. \\ \Rightarrow PQ = \frac{16}{\sqrt{5}}a = 20 \text{ m} \quad \therefore a = \frac{20\sqrt{5}}{16} = \frac{5\sqrt{5}}{4} \end{array}$$

$$c) \text{Area} = 2 \times \frac{1}{2} a^2 \int_0^\pi (3 + \sqrt{5} \cos \theta)^2 d\theta = a^2 \int_0^\pi 9 + 6\sqrt{5} \cos \theta + 5(\frac{1}{2} + \frac{1}{2} \cos 2\theta) d\theta$$

$$= a^2 \int_0^\pi \frac{23}{2} + 6\sqrt{5} \cos \theta + \frac{5}{2} \cos 2\theta d\theta = \frac{1}{2} a^2 \int_0^\pi 23 + 12\sqrt{5} \cos \theta + 5 \cos 2\theta d\theta$$

$$= \frac{1}{2} a^2 \left[23\theta + 12\sqrt{5} \sin \theta + \frac{5}{2} \sin 2\theta \right]_0^\pi = \frac{23\pi}{2} a^2 \quad \therefore \text{Area} = \frac{23\pi}{2} \times \frac{25 \times 5}{16}$$

$$= \frac{2875}{32} \pi \approx 282 \text{ m}^2$$

9. (a) The point P represents a complex number z in an Argand diagram. Given that

$$|z - 2i| = 2|z + i|,$$

(i) find a cartesian equation for the locus of P , simplifying your answer. (2)

(ii) sketch the locus of P . (3)

(b) A transformation T from the z -plane to the w -plane is a translation $-7 + 11i$ followed by an enlargement with centre the origin and scale factor 3.

Write down the transformation T in the form

$$w = az + b, \quad a, b \in \mathbb{C}. \quad (2)$$

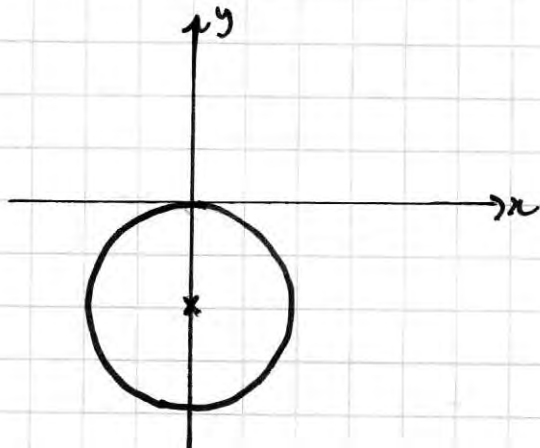
$$|x + (y-2)i| = 2|x + (y+1)i|$$

$$\Rightarrow x^2 + (y-2)^2 = 4(x^2 + (y+1)^2)$$

$$\Rightarrow x^2 + y^2 - 4y + 4 = 4x^2 + 4y^2 + 8y + 4$$

$$\Rightarrow 3x^2 + 3y^2 + 12y = 0 \Rightarrow x^2 + y^2 + 4y = 0 \quad x^2 + (y+2)^2 = 4$$

circle $C(0, -2) r=2$



$$w = 3z - 21 + 33i$$

10.

$$y \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx} \right)^2 + y = 0.$$

(a) Find an expression for $\frac{d^3 y}{dx^3}$.

(5)

Given that $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$,

(b) find the series solution for y , in ascending powers of x , up to and including the term in x^3 .

(5)

(c) Comment on whether it would be sensible to use your series solution to give estimates for y at $x = 0.2$ and at $x = 50$.

(2)

$$\frac{d}{dx} \left(y \frac{d^2 y}{dx^2} \right) + \frac{d}{dx} \left[\left(\frac{dy}{dx} \right)^2 \right] + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} \frac{d^2 y}{dx^2} + y \frac{d^3 y}{dx^3} + 2 \left(\frac{dy}{dx} \right) \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 0$$

$$y \frac{d^3 y}{dx^3} + 3 \left(\frac{dy}{dx} \right) \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 0$$

$$x_0 = 0 \quad y_0 = 1 \quad y'_0 = 1$$

$$y_0 y''_0 + (y'_0)^2 + y_0 = 0 \Rightarrow y'' = -2$$

$$y y'''_0 + 3(y'_0) y''_0 + y'_0 \Rightarrow y''' = 5$$

$$\therefore y = 1 + x - x^2 + \frac{5}{6} x^3 - \dots$$

c) fine for $x = 0.2$ but not fine for $x = 50$ as approximation will be divergent for large values of x